BULLETIN DE L'ACADÉMIE POLONAISE DES SCIENCES
Série des sciences math., astr. et phys. — Vol. XXIV, No. 4, 1976

MATHEMATICS
(GENERAL TOPOLOGY)

A Note on Lower Semicontinuous Set-valued Maps *)

by

KA-SING LAU

Presented by K. KURATOWSKI on April 10, 1975

Summary. Let X be a topological space, K a compact subset of a locally convex space Y and $c(K)$ the family of closed convex subsets of K. It is shown that a map $\Phi : X \to c(K)$ is lower semicontinuous if the set $\{x : \Phi(x) \cap H \neq \emptyset\}$ is open for any $H = \{x : f(x) > r\}$ where f is a continuous functional on Y. A simple example in R^2 shows that the compactness assumption on K is essential

1. Introduction. Let X, Y be topological spaces. We use 2^Y to denote the family of closed subsets of Y. A set-valued map $\Phi : X \to 2^Y$ is called lower semicontinuous if for any open set U in Y, the set $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X. It is well known that the lower semicontinuous set-valued maps play a significant role in the continuous selection theories (cf. [3, 4]). If Y is a locally convex (Hausdorff) linear topological space and K is a closed subset of Y, we let $c(K)$ denote the family of closed convex subsets of K. A set-valued map $\Phi : X \to 2^Y$ is called *weakly lower semicontinuous* if the set $\{x \in X : \Phi(x) \cap H \neq \emptyset\}$ is open for any open half space $H = \{y \in Y : f(y) > r\}$, where $f \in Y^*$, $r \in R$. Our main purpose is to prove

THEOREM 1. *Suppose X is a topological space, Y a locally convex space and K a compact subset of Y. Let $\Phi : X \to c(K)$ be a set-valued map. Then Φ is lower semicontinuous if and only if it is weakly lower semicontinuous.*

Although the given topology and the $w(Y, Y^*)$ topology coincide in K, it is not immediate that the sets $\{x \in X : \Phi(x) \cap H_i \neq \emptyset\}$, $i = 1, \ldots, n$ are open, will imply that $\{x \in X : \Phi(x) \cap \bigcap_{i=1}^{n} H_i \neq \emptyset\}$ is open. Hence one side of the theorem is non--trivial.

2. Proof of the theorem. Theorem 1 will result from the following lemmas. In the proofs, we will make use of the Vietoris topology [1]. Let X be a topological space. A subbase for the *Vietoris topology* on 2^X consists of all sets having one of the following forms:

$$\{F \in 2^X : F \cap U \neq \emptyset\}, \quad \{F \in 2^X : F \subseteq U\},$$

where U is an arbitrary open set in X.

---

*) Revised version received on August 5, 1975.

LEMMA 2. *Let X be a topological space and let 2^X be given the Vietoris topology, then*

(i) *if X is compact, so is 2^X.*

(ii) *if X is regular, let $\{F_\alpha\}_{\alpha\in I}$ be a net in 2^X converging to F_0, then for each $x\in X$, we have an equivalence: $x\in F_0$ if and only if there exists a net $\{x_\alpha\}_{\alpha\in I}$, $x_\alpha\in F_\alpha\ \forall\ \alpha\in I$, converging to x in X.*

Proof. (i) follows from Theorem 15 in [1]. Suppose the necessity part of (ii) were not true, there exist a subnet $\{F_\beta\}_{\beta\in J}$ of $\{F_\alpha\}_{\alpha\in I}$ and an open neighborhood U of x such that $F_\beta\cap U=\emptyset$ for each β. Note that $\{F_\beta\}_{\beta\in J}$ converges to F_0. The family

$$\mathscr{J}=\{F\in 2^X\colon F\cap U=\emptyset\}$$

is a closed subset in 2^X and $F_\beta\in\mathscr{J}$ for all $\beta\in J$. But $F_0\notin\mathscr{J}$ (for $x\in F_0\cap U$), a contradiction. The sufficiency follows immediately from the definition of the Vietoris topology and the regularity of the space X.

LEMMA 3. *Let X be a subset of a locally convex space and let $c(X)$ be the family of closed convex subsets of X, then $c(X)$ is closed in 2^X.*

Proof. Let $\{F_\alpha\}_{\alpha\in I}$ be a net in $c(X)$ converging to F_0. We only need to show that F_0 is convex. Consider $\lambda x+(1-\lambda)y$, $x, y\in F_0$, $0<\lambda<1$. By Lemma 2 (ii), there exist two nets $\{x_\alpha\}_{\alpha\in I}$, $\{y_\alpha\}_{\alpha\in I}$, $x_\alpha, y_\alpha\in F_\alpha\ \forall\alpha\in I$, converging to x, y, respectively. Since F_α is convex, $\lambda x_\alpha+(1-\lambda)y_\alpha$ is in F_α for each $\alpha\in I$. That $\{\lambda x_\alpha+(1-\lambda)y_\alpha\}_{\alpha\in I}$ converging to $\lambda x+(1-\lambda)y$ and Lemma 2 (ii) imply that $\lambda x+(1-\lambda)y$ is in F_0. Hence F_0 is convex.

Our key step is to prove the following proposition.

PROPOSITION 4. *Let Y be a locally convex space, K a compact subset of Y, $y_0\in K$ and U an open neighborhood of y_0 in Y. Then there are open half spaces $H_1, \dots, H_n$ in Y containing y_0 such that every closed convex subset $S\subseteq K$ which intersects $H_1, \dots, H_n$ must intersect U.*

Proof. Let $D=K\setminus U$. Then D is compact, and so is 2^D with the Vietoris topology. By Lemma 3, $c(D)$ is closed and hence compact in 2^D. Let $\mathscr{H}$ be the collection of open half spaces in Y containing y_0. Since Y is locally convex, by the separation theorem, each F in $c(D)$ is a subset of $Y\setminus\bar{H}$ for some $H\in\mathscr{H}$. Hence the sets

$$\mathscr{U}_H=\{F\in 2^D\colon F\subseteq Y\setminus\bar{H}\},\qquad H\in\mathscr{H}$$

form an open cover of $c(D)$. There exists a finite subcover $\mathscr{U}_{H_1}, \dots, \mathscr{U}_{H_n}$. These $H_1, \dots, H_n$ satisfy the requirement. Indeed, if S is a closed convex subset in K such that $S\cap U=\emptyset$, then $S\subseteq D$ and $S\in\mathscr{U}_{H_i}$ for some $i=1, \dots, n$. This implies that $S\cap H_i=\emptyset$ for some $i=1, \dots, n$.

Proof of Theorem 1. The necessity is clear. To prove the sufficiency, it is enough to prove that for any open set U in Y such that $\Phi(x_0)\cap U\neq\emptyset$, the set $\{x\in X\colon \Phi(x)\cap U\neq\emptyset\}$ is a neighborhood of x_0. Let $y_0\in\Phi(x_0)\cap U$ and let

$H_1, \ldots, H_n$ be the open half spaces constructed in the above proposition. Since each $\Phi(x)$ is closed and convex, it follows that

$$\bigcap_{i=1}^{n} \{x \in X : \Phi(x) \cap H_i \neq \emptyset\} \subseteq \{x \in X : \Phi(x) \cap U \neq \emptyset\}.$$

By assumption each set on the left side is an open neighborhood of x_0, hence so is $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$.

3. Remarks. Combining Theorem 1 and the Michael selection theorem, we have

COROLLARY 5. *Suppose X is a compact Hausdorff space, Y a metrizable locally convex space and K a compact subset of Y. Let $\Phi : X \to c(K)$ be a weakly lower semicontinuous map and let f be a continuous function defined on a closed subset F in X with values in Y and such that $f(x) \in \Phi(x)$ for each $x \in F$. Then f can be extended to a continuous function $\bar{f}$ on X such that $\bar{f}(x) \in \Phi(x)$ for each $x \in X$.*

An application of this corollary is shown in [2]. We finally remark that Theorem 1 will not be true if we do not assume that each of the $\Phi(x)$ is contained in a compact subset of Y. Consider the map Φ from $X = [0, 1]$ into $c(R^2)$ with $\Phi(0) = \{(1, y_2) : y_2 \in R\}$ and $\Phi(x) = \{(y_1, y_2) : x \cdot y_2 = y_1,\ y_1, y_2 \in R\}$ for $x \neq 0$. Then Φ is not lower semicontinuous. But for any open half space H in R^2, the set $\{x \in X : \Phi(x) \cap H \neq 0\}$ is either $[0, 1]$ or $[0, 1] \setminus \{x'\}$ for some x' in X, hence Φ is weakly lower semicontinuous. If $Y = R$, the two conditions will be equivalent even without the compactness condition. For in this case we have

$$\{x \in X : \Phi(x) \cap (a, b) \neq \emptyset\} = \{x \in X : \Phi(x) \cap (-\infty, b) \neq \emptyset\} \cap \cap \{x : \Phi(x) \cap (a, \infty) \neq \emptyset\}$$

for any a, b in R with $a < b$.

UNIVERSITY OF PITTSBURGH, PITTSBURGH, Pa. 15260 (U.S.A.)

REFERENCES

[1] O. Frink, Jr., *Topology in lattices*, Trans. Amer. Math. Soc., **51** (1942), 569—582.

[2] K. Lau, *Extreme operators on Choquet simplexes*, Pacific J. Math., **52** (1974), 129—142.

[3] E. Michael, *Continuous selections* I, Annals of Math., **63** (1956), 361—382.

[4] T. Parthasarathy, *Selection theorems and their applications*, Springer Verlag, Lecture Notes in Mathematics, No. 263 (1972).

Ка-Синг Лау, **Заметка о полунепрерывных снизу многовалентных преобразованиях**

Содержание, Пусть X будет топологическим пространством, K — компактным множеством локально выпуклого пространства Y, с (K) — семейством замкнутых выпуклых подмножеств. Докажем, что преобразование $\Phi : X \to c(K)$ есть полунепрерывное снизу если множество $\{x : \Phi(x) \cap H \neq \emptyset\}$ открытое для любого $H = \{x : f(x) > r\}$ж где f является непрервным функционалом на Y. Простой пример в R^2 показывает, что предположение компактности K существенно.